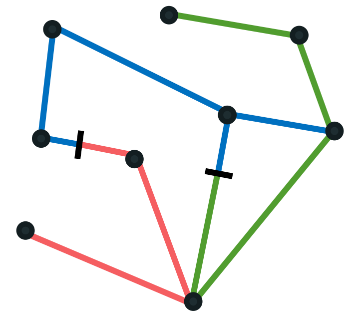

Approximate Envy-Freeness in Graphical Cake Cutting

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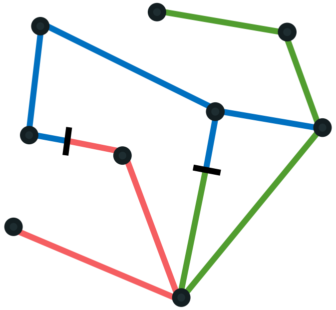
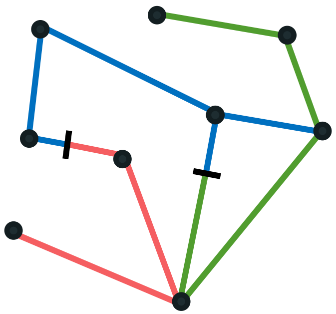
Cake Cutting



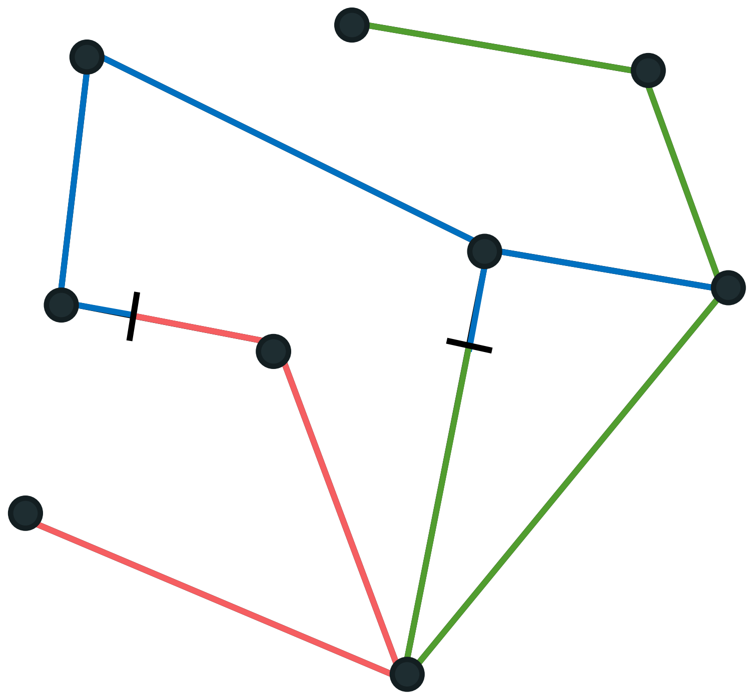
- The problem of fairly allocating a divisible resource.
- Typical assumption: cake is represented by an interval [0, 1].
  - Appropriate when the resource is linear.
  - Insufficient when the resource is a network.
- A more general model: Graphical Cake Cutting†

† Dividing a Graphical Cake (Bei and Suksompong, 2021)

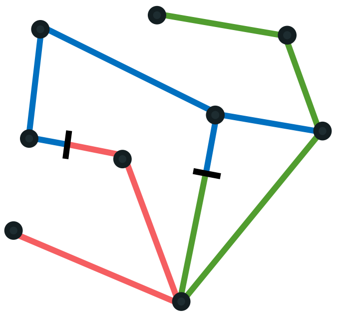
Graphical Cake Cutting



- Resource is represented by a connected graph.
- Cake lies on the edges of the graph, can be subdivided.
- Each agent receives a connected piece of the graph.



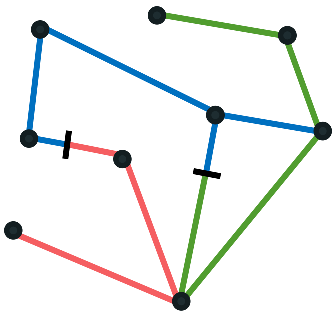
Fairness Notions



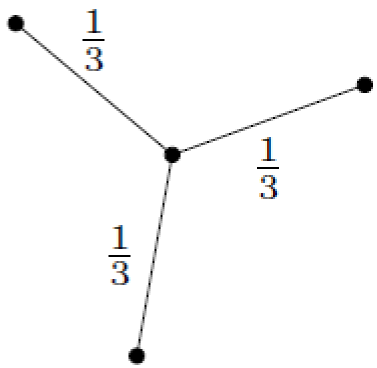
- Common fairness notions are proportionality, maximin share, and envy-freeness.

- Graphical cake cutting:
    - Proportionality (Bei and Suksompong, 2021)
    - Maximin share (Elkind, Segal-Halevi and Suksompong, 2021)
    - Envy-freeness (our work)
        - No agent would rather have another agent's piece of cake.

Envy-Freeness

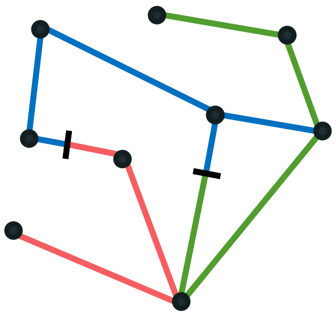


- An envy-free allocation may not exist.



- We consider approximations of envy-freeness.
  - α-additive-EF
    - agent *i* does not envy agent *j* by an amount of more than $0 \leq \alpha \leq 1$.
  - α-EF
    - agent *i* does not envy agent *j* by a factor of more than $\alpha \geq 1$.
- **Proposition:** α-EF implies $(\alpha - 1)/(\alpha + 1)$-additive-EF.

Our Results



	General graphs	Star graphs
Non-identical valuations	1/2-additive-EF	(3 + ε)-EF
Identical valuations	(2 + ε)-EF	2-EF

- Our work presents efficient algorithms for computing these allocations.

General Graphs, Non-Identical Valuations

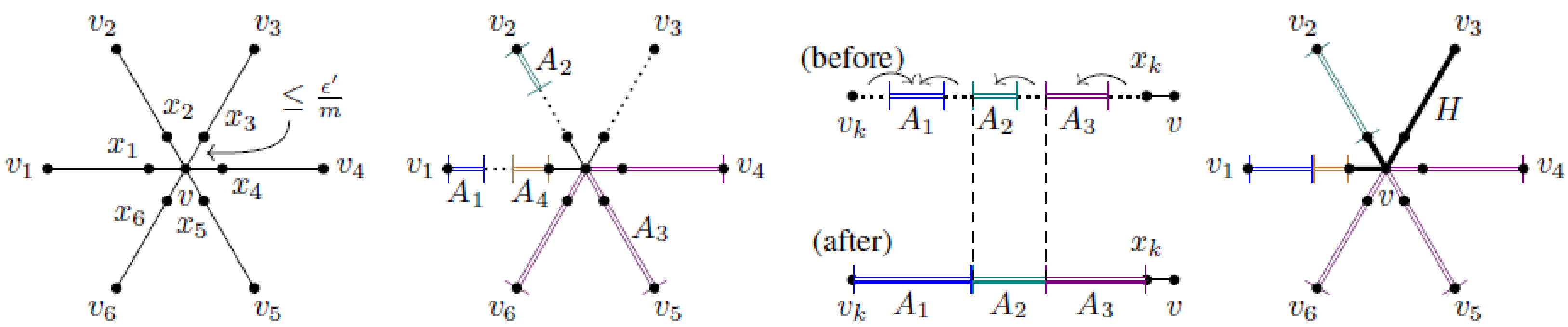
- **Theorem:** There exists a 1/2-additive-EF allocation for a general graph.
- **Proof sketch:** Generalize ideas from interval cake cutting.†
  - Find a bundle worth less than 1/2 to every agent and at least 1/4 to some agent; allocate to the agent who values it at least 1/4.
    - To find this bundle, use the DIVIDE algorithm‡ with threshold 1/4.
  - Repeat the procedure with the remaining graph and remaining agents.
  - This gives a 1/2-additive-EF allocation.

	General graphs	Star graphs
Non-identical valuations	1/2-additive-EF	$(3 + \varepsilon)$-EF
Identical valuations	$(2 + \varepsilon)$-EF	2-EF

† Contiguous Cake Cutting: Hardness Results and Approximation Algorithms (Goldberg, Hollender, and Suksompong, 2020)
‡ Dividing a Graphical Cake (Bei and Suksompong, 2021)

Star Graphs, Non-Identical Valuations

- **Theorem:** There exists a (3 + ε)-EF allocation for a star graph.
- **Proof sketch:** Generalize ideas from interval cake cutting.[†]



(a) Phase 1 (b) Phase 2 (c) Phase 3 (d) Phase 4

  - This gives a (3 + ε)-EF allocation.

	General graphs	Star graphs
Non-identical valuations	1/2-additive-EF	(3 + ε)-EF
Identical valuations	(2 + ε)-EF	2-EF

[†] Fair and Efficient Cake Division with Connected Pieces (Arunachaleswaran, Barman, Kumar, and Rathi, 2019)

General Graphs, Identical Valuations

- **Theorem:** There exists a (2 + ε)-EF allocation for a general graph and agents with identical valuations.

- **Proof sketch:** Recall that we used a threshold of 1/4 previously.
  - Use an adaptive threshold of $\frac{1}{2}\left(\frac{2i}{2n-1} - \sum_{j=1}^{i-1} \mu(A_j)\right)$ for the i^{th} agent; this gives a 4-EF allocation.
  - Next, generalize ideas from partitioning indivisible edges of a graph.†
  - Start with a 4-EF allocation and adjust adjacent bundles repeatedly; this gives a (2 + ε)-EF allocation.

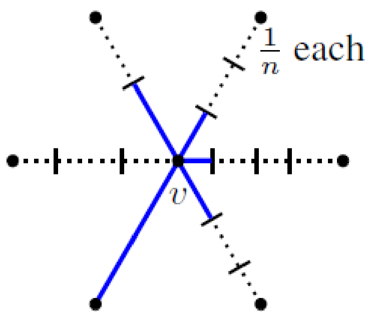
	General graphs	Star graphs
Non-identical valuations	1/2-additive-EF	(3 + ε)-EF
Identical valuations	(2 + ε)-EF	2-EF

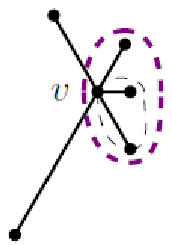
† A Tight Bound on the Min-Ratio Edge-Partitioning Problem of a Tree (Chu, Wu, Wang, and Chao, 2010)

Star Graphs, Identical Valuations

- **Theorem:** There exists a 2-EF allocation for a star graph and agents with identical valuations.

- **Proof sketch:** Use a bag-filling algorithm.

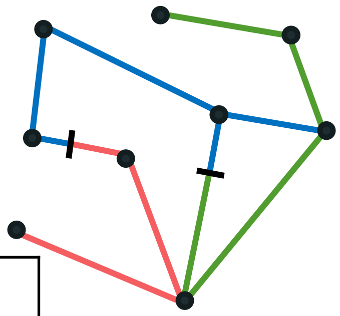




  - This gives a 2-EF allocation.

	General graphs	Star graphs
Non-identical valuations	1/2-additive-EF	$(3+\varepsilon)$-EF
Identical valuations	$(2+\varepsilon)$-EF	2-EF

Conclusion



- **Summary**
	General graphs	Star graphs
Non-identical valuations	1/2-additive-EF	$(3 + \varepsilon)$-EF
Identical valuations	$(2 + \varepsilon)$-EF	2-EF
  - The bounds for identical valuations are tight due to prior work.
  - **Open question:** Is an α-EF allocation guaranteed to exist for general graphs and non-identical valuations, for some constant $\alpha \geq 1$?

- **Omitted results**
  - Settings where agents can receive a small number of connected pieces: use the notion of path similarity number.

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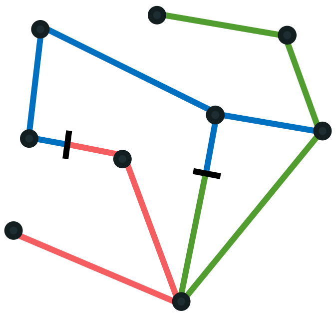

Thank you!

Questions?